Relations and Functions

Course: CS205 M

Lecture 3

https://knmadnani.github.io/Courses/TCSMn.html

What is a Relation?

- A relation R from set A to set B is a subset of $A \times B$.
- Example: $A = \{1, 2\}$, $B = \{x, y\}$, $R = \{(1, x), (2, y)\}$
- Relations can represent mappings, orderings, etc.

Types of Relations

- Reflexive: $(a, a) \in R$ for all $a \in A$
- Symmetric: $(a, b) \in R$ implies $(b, a) \in R$
- Transitive: $(a, b) \in R$ and $(b, c) \in R$ implies $(a, c) \in R$
- Antisymmetric: $(a, b) \in R$ and $(b, a) \in R$ implies $a = b$

Representation of Relations

- Set of ordered pairs
- Matrix representation
- Graph representation (digraphs)
- Example: $R = \{(1, 2), (2, 3)\} \rightarrow$ Digraph with edges

Equivalence Relations

- A relation that is reflexive, symmetric, and transitive
- Partitions the set into disjoint equivalence classes.
- Example: Congruence modulo n on integers

Partial Order Relations

- Reflexive, antisymmetric, transitive
- Partially ordered set (poset)
- Example: Subset relation $\subseteq$ on power set

What is a Function?

- A function f from A to B assigns exactly one element of B to each element of A
- Notation: $f: A \rightarrow B$
- Example: $f(x) = x^2$, $A = \{1, 2, 3\}$, $B = \{1, 4, 9\}$

Types of Functions

- One-to-one (Injective): $f(a) = f(b) \Rightarrow a = b$
- Onto (Surjective): $\forall\, b \in B,\ \exists\, a \in A$ such that $f(a) = b$
- Bijective: Both injective and surjective
- Inverse of a function is, in general, a relation.
- Inverse of a function f is a function iff f is bijective.

Function Composition

- If $f: A \to B$ and $g: B \to C$, then $g \circ f: A \to C$
- $(g \circ f)(x) = g(f(x))$
- Associative but not necessarily commutative.
- Fix point?

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- Fix point?
- $f(f(f(f(f(f(x)))))) = f^6(x)$.
- y is a fix point of $f(x)$ iff $f^n(x) = f^{n+1}(x)$

Interesting Puzzle 1

- Let f: $\mathbb{Z} \rightarrow \mathbb{Z}$ be defined as:
- For n>1, f(n) = n/2 if n is even, and 3n+1 if n is odd. f(1) = 1.
- Question: What happens to f when iterated repeatedly?
- Does this function reach a fix point for any value of n?
- Collatz Conjecture: For any number n, the above function reaches a fixpoint i.e. 1. Still a conjecture. Not yet proven.